On the Inertial Stability of the Equatorial Middle Atmosphere[1]

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(Manuscript received 19 March 1981, in final form 22 June 1981)

ABSTRACT

A theory of inertial instability on the equatorial beta-plane is developed with application to the inertial stability of the equatorial middle atmosphere at the solstices. It is shown that the stability of this region depends primarily on two unknowns. First, there is the question of whether eddy diffusion can be regarded as stabilizing, or whether this diffusion actually arises from the instability itself. Second, because the diabatic circulation would appear to induce a cross-equatorial shear much greater than that observed, or than that modeled in Holton and Wehrbein (1980), it appears that the gravity wave-induced decelerations would be crucial to the stability of this flow. Unfortunately, the parameterization scheme of Leovy (1964) designed to mimic this effect obscures the issue, since this "frictional drag" concept is invalid on a local basis (Lindzen, 1981).

The expected structure and vertical wavelength of the equatorial inertial instability is discussed in the context of this simple model. Predicted vertical wavelengths also depend on the unknown factors listed above. The greatest likelihood of an observable inertial instability would be in the winter tropical mesosphere, within 10° of the equator.

1. Introduction

For a number of years there has been recognized the existence of inertial instability in rotating fluids which arises from an imbalance of pressure gradient and centrifugal forces when the absolute value of angular momentum decreases with radius (Rayleigh, 1916). This instability, like its counterpart, convection, can be understood as a "parcel" instability; i.e., it is due to an imbalance of local forces acting on individual fluid parcels. This is quite different from the overreflecting wave instabilities known as barotropic, baroclinic and stratified shear instability (Lindzen and Tung, 1978). On the spherical earth the inertial instability may be due to a reversal in the sign of the relative vorticity of the flow, a criterion expressed mathematically as

$$f(f - \bar{u}_y) < 0, \tag{1.1}$$

where f is the Coriolis parameter and $\bar{u}_y$ is the horizontal wind shear. This is a modified form of Rayleigh's criterion taking into account Coriolis forces in the rotating reference frame for motions which are quasi-horizontal. A further modification takes into account vertical wind shear

$$1 - \mathrm{Ri}^{-1} - \bar{u}_y/f < 0, \tag{1.2}$$

where Ri is the Richardson number $N^2/\bar{u}_z^2$. In this case the latitudinal pressure gradients adiabatically ensuing from vertical displacements help to destabilize the flow; in fact, when Ri < 1, horizontal shear is no longer an essential ingredient in the instability. It is desirable to mention that in this case the criterion seems very significant, in view of the fact that the corresponding criterion for wavelike shear instability is Ri < 1/4 (Miles, 1961). Perhaps for this reason, inertial instability has been the subject of a recent study relating to organized mesoscale convective systems (Emanuel, 1979).

A somewhat underappreciated aspect of the theoretically-predicted inertial instabilities is their singularity with respect to diffusion. Because preferred inviscid instabilities often have vanishing length scales (e.g., Stone, 1966), it ultimately becomes necessary to incorporate at least a small diffusion in the equations (Kuo, 1954; McIntyre, 1970). It follows that not only does diffusion select the scale of the most unstable motion, but it also alters the instability criterion itself, at least by a small amount. Another interesting result of diffusion is the possibility of "overstability" when the Prandtl number differs from unity (McIntyre, 1970). For further comment on these and other historical aspects of the subject, the reader is referred to Emanuel (1979, Section 2).

In this paper, we will address what is beginning to be appreciated as a new and possibly significant problem in the dynamics of the middle atmosphere. The question of interest concerns the inevitable

[1] Contribution No. 599, Department of Atmospheric Sciences, University of Washington, Seattle.

0022-4928/81/112354-11$06.75

inertial instability of horizontal shear at the equator, which itself results from the diabatic solstice circulation from summer to winter hemisphere (Holton and Wehrbein, 1980). For the most part, we will treat the problem conceptually as a horizontal, Rayleigh angular momentum instability (in contrast to the current interest in tropospheric mesoscale vertical shears). According to (1.1), *any* horizontal shear near the equator is inertially unstable. From a theoretical standpoint, it is immediately of interest to examine the stability properties as affected by diffusion and other forms of dissipation. From an observational standpoint, it seems remarkable that the inertial instability of this region has been overlooked in the literature.[2] Nevertheless, this paper will attempt to determine the stability of the equatorial middle atmosphere near the equator by utilizing the theoretical criteria in conjunction with the commonly accepted forms of dissipation. It will be shown that the stability of this region depends on the stabilizing influence of eddy diffusion, and upon the deceleration caused by internal gravity waves.

The theoretical development beginning in Section 2 below therefore involves discussion of the instability of mean zonal flows with nonzero horizontal shear at the equator. Because of the strong variation of f with latitude, it seems more appropriate to address the inertial stability problem with the equatorial β-plane concept (Lindzen, 1967) in which the Coriolis parameter is a linear function of the distance from the equator. While there are undoubtedly problems which demand the alternate, local f-plane approach (such as those involving the inertial instability of primarily vertical shears near the equator, as discussed in Appendix C of Emanuel, 1979) the β-plane approach seems well-suited here. Interestingly, from a mathematical standpoint the beta-plane approach (in contrast to its f-plane counterpart) automatically ensures latitudinal boundedness, even though we formally regard all motions as "unbounded", i.e., with boundary conditions imposed at infinity. Boundedness immediately leads to important results dealing with diffusive scale selection and flow stabilization.

For the β-plane problem discussed here some important additional background comes out of Boyd (1978) who discussed a "γ-plane" approximation used for modeling equatorial waves in horizontal shear. The γ-plane perturbation technique employs the small parameter defined by the ratio of latitudinal to zonal scales, a method which, as shown by Boyd, is very helpful in understanding the behavior of ultralong Rossby-gravity waves, for example, in horizontal linear and parabolic shear. For our purposes it is noteworthy that the approximate, lowest order γ-plane streamfunction equation is identical to the *exact* equation for symmetric inertio-gravity disturbances. Also, the presence of a linear vertical shear results in only a slight complication, yielding an equation of the same form. The resulting eigenfunctions are the familiar Hermite polynomials modified only by the (linear or parabolic) shear, together with a dispersion relation similar to that found for equatorial waves on the β-plane. The key to understanding symmetric instability is to employ this relation in reverse, that is, by searching for imaginary frequencies resulting in wave growth. As expected, imaginary roots always exist in the inviscid problem when the horizontal shear is nonzero at the equator.

Also anticipated is the scale selection and flow stabilization when diffusion is formally included in the equations. These results suggest that when a second-order diffusion is included, there exists a critical horizontal shear for instability, dependent on the one-fifth power of the diffusion coefficient. By hypothesizing that *eddy* diffusion is of this mathematical form, we can employ the second-order diffusion test for inertial stability in conjunction with the observed and model shears, and the estimated values of eddy diffusion. As already remarked, it is found that the inertial stability of this region depends on whether or not this eddy diffusion can be regarded as stabilizing, and on the deceleration of the mesospheric zonal mean flow by vertically-propagating internal gravity waves (Lindzen, 1981).

Although the author's interest in this subject arose largely from the theoretical and observational questions associated with it, it was learned in the late stages of this investigation that an inertial instability of the sort to be discussed below has been observed in the recent numerical work of Hunt (1981). Therefore, Hunt's paper serves as complimentary to this work, and affords an additional reference of benefit to the reader.

Here, we first direct attention to a linear, β-plane theory of the inertial instability in Section 2, followed by an application of these ideas to the middle atmosphere in Section 3.

2. Inertial instability on the equatorial β-plane

Following Emanuel (1979), we consider linear, axisymmetric perturbations on a hydrostatically and geostrophically balanced zonal mean flow having constant static stability and constant linear shear. The equatorial β-plane equations have appeared frequently in the literature (e.g., Boyd, 1978). In effect we set the zonal wavenumber equal to zero

[2] *Note added in proof*: The baroclinic equatorial inertial stability problem was, however, treated by P. H. Stone, 1971: The symmetric baroclinic instability of an equatorial current. *Geophys. Fluid Dyn.*, **2**, 147–164.

in consideration of symmetric disturbances. This greatly simplifies matters, just as in the γ-plane approximation, because three otherwise difficult effects (Doppler shift, longitudinal pressure gradient and longitudinal divergence) do not appear in the equations. As a result of this simplification, the motion becomes nondivergent in the meridional plane, thereby defining a streamfunction ψ; furthermore, it is possible to incorporate a linear vertical shear in the problem without adding undue complication. During the course of the following discussion, we will have occasion to discuss the effect of linear shear in height; however, since the primary thrust of this paper deals with horizontal shear, vertical shear will for the most part be set to zero. By way of caution it should be remarked that probably the equatorial β-plane approach is not generally suited to discussing inertial instability due to vertical shear only, although it can clearly reveal the effect of a vertical shear on what is fundamentally a horizontal instability. One reason for this is a breakdown in the geostrophic approximation near the equator, which is otherwise required to relate thermal to momentum effects. While even this assumption is valid for zonal mean flows, it clearly is not valid on a more local basis, as might be the case when dealing with locally strong vertical shears, for example. Fortunately, vertical shear in the middle atmosphere is almost always extremely small (except in the descending westerlies of the semiannual and quasi-biennial oscillations) compared to the static stability.

Symmetric disturbances on the equatorial β plane satisfy the streamfunction equation

$$\psi_{yy} - \frac{m^2}{N^2}[\beta y(\beta y - \gamma) - \omega^2]\psi = 0, \quad (2.1)$$

where m, ω and γ are vertical wavenumber, frequency, and horizontal shear, respectively, and $\beta = df/dy$. This result corresponds to Boyd's (1978) discussion of the effect of linear shear on the Rossby–gravity wave at lowest order in the γ-plane approximation. In our case, however, it is an exact equation for a purely symmetric disturbance.[3] As discussed by Boyd, (2.1) reduces to the canonical form of the harmonic oscillator equation in the usual manner (e.g., Holton, 1975, p. 64) except that one must first shift the latitude in the direction of the horizontal shear, *viz.*,

$$y_1 = y - \gamma/2\beta. \quad (2.2)$$

With respect to this shifted latitude, the streamfunction solutions are the orthogonal set of Hermite polynomials multiplied by a Gaussian. The lowest-order mode is simply a Gaussian, and is most relevant to our present purposes (as it will appear to have the largest growth rate). The latitudinal scale factor is just ω_1/β, where

$$\omega_1^2 = \omega^2 + \gamma^2/4. \quad (2.3)$$

The eigencondition for this lowest mode, however, is

$$\omega_1^2 = N\beta/|m|. \quad (2.4)$$

Now in discussing forced waves and how linear shear affects them, we would regard ω_1 as an enhanced "effective" frequency, and regard it as being determined by ω and γ while, in turn, defining m^2/N^2 as an eigenvalue. However, in discussing instability, which is analogous to discussing "free" modes, ω_1 may be regarded as given, thereby determining the frequency. It is apparent from comparison of (2.3) and (2.4) that, for a given γ, the square of the frequency can be minimized by increasing m. As m is made indefinitely large, the frequency becomes imaginary, and approaches the asymptotic value $-\frac{1}{2}\gamma i$. The fastest growing mode in this inviscid problem is therefore of simplest structure ($n = 0$), and of infinite vertical wavenumber (m) and vanishing horizontal scale (y_0). Growing modes always exist for any nonzero shear although, of course, their growth rates are limited by this shear. In accord with the f-plane theory, the magnitude of the growth rate also can be regarded as equal to the value of the Coriolis parameter in the center of the unstable region bounded below by the equator (for positive shear) and above by the latitude $\beta y = \gamma$.

a. *Diffusive scale selection and flow stabilization*

Because the preferred mode of instability on the equatorial β-plane has essentially infinite vertical wavenumber, it is necessary to incorporate a scale-dependent dissipation into the problem in order to arrive at a finite scale of maximum growth. It is the purpose of this and the following subsection to display how unstable mode scales are selected for various kinds of dissipation, and also to show how these affect the instability criterion itself. Here, we will focus attention on the most obvious form of scale-dependent dissipation, *viz.*, diffusion. For the purposes of analytical simplicity we restrict the dissipation to being dependent only on vertical wavenumber; otherwise the order of the eigenproblem would be unduly increased.

The most obvious case involves the assumption of a Prandtl number equal to unity, where

$$\sigma = \nu/\kappa, \quad (2.5)$$

[3] In response to a question raised by a referee, it should be noted that a term proportional to the horizontal component of the Coriolis force (f^*) is neglected here. However, this term is isomorphic to the vertical shear term [see (2.8)], and is as small as $(f^*/N)^2$.

with ν and κ being the diffusivity of momentum and heat, respectively. When $\sigma = 1$ the diffusion alters the frequency as $\omega \rightarrow \omega - i\nu m^2$. Consequently, our indefinite increasing of m in the inviscid problem, when applied here, no longer results in imaginary frequencies in that limit. Rather, there exists a single minimum in ω^2 which defines a point of maximum growth. Evidently, when this point itself lies on the real ω axis, we have determined the wavenumber of instability onset, together with a critical shear, below which the flow is stable. One can, in fact, determine this critical wavenumber m_c and shear γ_c from dimensional considerations alone, since both depend only on ν and $(N\beta)$. The resulting formulas turn out to be very accurate since the exact expressions are

$$m_c^5 = N\beta/4\nu^2, \tag{2.6}$$

$$|\gamma_c| = 2(5)^{1/2}\nu^{1/5}(\tfrac{1}{4}N\beta)^{2/5}, \tag{2.7}$$

Of particular interest is the weak dependence of both quantities on the diffusion—a fact which should comfort most observationalists in knowing that our ignorance of the magnitude of diffusion in the middle atmosphere can be partly excused—for example, a variation in diffusion of two orders of magnitude would alter γ_c by only 2.5 times.

When the diffusion is equal to unity in MKS units, for example, the predicted horizontal and vertical scales in the middle atmosphere are approximately 5.25° and 2.4 km, respectively. Hence Emanuel's statement (1979, p. 2438) to the effect that disturbance scales are more related to the depth of the unstable region, rather than the viscous scale, except for very small diffusion, is not seen to be directly pertinent to our problem. Here predicted scales would be basically viscous, except for very large diffusion.

A schematic diagram of the neutral mode is given in Fig. 1. The streamfunction and meridional velocity are symmetric about the shifted equator, while the vertical velocity and temperature are antisymmetric about this point. On the other hand, the zonal velocity is shifted beyond the shifted equator, and is asymmetric about the boundary of the unstable region, so that it appears distorted (its Gaussian being still centered on the shifted equator). For each field, the generation mechanisms are everywhere being balanced by diffusion out of the maxima.

While the scale of display in Fig. 1 is arbitrary, it should be remembered that in reality the aspect ratio of the disturbance depends weakly on the diffusion, with the ellipses flattening out as the diffusion goes to zero.

There are two variations on the above problem which might be of interest in some contexts, though are not deserving of a considerable space here. The first involves the inclusion of vertical shear. This enters entirely *via* the Richardson number Ri, and is important in several respects. First, reduction to the harmonic oscillator equation requires the substitution

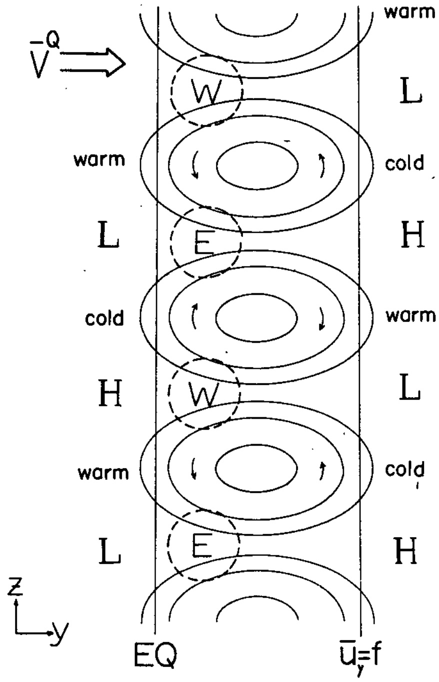

FIG. 1. The lowest-order neutral viscous mode on the equatorial β plane. The diabatic circulation $\bar{v}^Q$ induces a linear shear which, at some critical value, generates an unstable region bounded below by the equator, and above by the latitude $\bar{u}_y = f$.

$$\psi(y) = \chi(y)\exp(-\tfrac{1}{2}\beta y^2 im\bar{u}_z/N^2) \tag{2.8}$$

which in effect provides a varying disturbance orientation in latitude, the very same kind of preferred (but constant) orientation one finds in the f-plane problem. Second, the resulting χ equation has a term $(1 - \mathrm{Ri}^{-1})$ multiplying $\beta^2 y^2$. Unbounded solutions exist only when $\mathrm{Ri} > 1$ (analogous to what happens to the Rossby–gravity wave on the γ-plane when the mean flow becomes barotropically unstable); when the Richardson number falls below unity, presumably we need then to consider the finite extent of the flow. In any case, it is found that the finite Richardson number is destabilizing as it acts to reduce the critical shear (and the corresponding vertical wavenumber).

The second variation considers a Prandtl number which differs from unity. There is found to be a weak increasing dependence of γ_c on σ; however, when $\sigma \gtrsim \tfrac{3}{2}$, it appears that "overstable" modes, i.e., those with nonzero real frequency component, have critical shears slightly lower than the corresponding nonoscillatory instabilities. While this is a very interesting piece of information, at the present time one cannot make much use of it in view of the uncertainty in assessing real values of Prandtl number in the middle atmosphere.

b. Higher-order diffusion

One limitation of the formulas (2.6) and (2.7) is their being restricted to a diffusion which is specifically second order. While this is the usual form of molecular diffusion, it is not necessarily correct for *eddy* diffusion, although one commonly sees it employed even in this connection.

Under more general circumstances we might envision a diffusive process arising from turbulence, but of a higher order than the second-order process described above. Supposing for the sake of argument that this is an infinite-order diffusion, the question arises as to how the stability criterion is affected.

Actually, this is the simplest case to investigate, since the instability criterion would then originate in (2.3) and (2.4). It was there found that in order to have growing modes one must definitely reduce the vertical wavelength below some critical vertical wavelength (call it λ_t). The latter would vary as the square of the cross-equatorial shear γ. Since in an infinite-order diffusion the net effect would be to suppress all scales of motion below some critical vertical scale, the instability criterion would then be that this latter scale be less than the necessary vertical wavelength for instability λ_t.

Now while this stability criterion does not necessarily save us any effort, since finding a critical vertical scale may be as difficult as a precise definition of the eddy diffusion coefficient, it does provide an additional subjective test of the inertial stability of a flow. In Section 3, both the second-order diffusion test and the more subjective test suggested here will be applied to the equatorial middle atmosphere.

3. The stability of the equatorial middle atmosphere

Having discussed the inertial instability problem in a theoretical way in the preceeding section, with an emphasis on the dissipative processes relevant to it, the next step is to apply the various criteria to the equatorial middle atmosphere at and around the solstices, when a significant horizontal shear supposedly exists at the equator. In the investigation of this subject, however, the author was confronted with a series of difficult and enigmatic questions, which seemed to preclude a certain answer to the question of the inertial stability of this region. However, with the theory of Section 2 in hand, we will find that it is at least possible to construct hypothetical answers to these questions, while at the same time shedding light on the individual issues involved. It is the goal of this section to do precisely that, and the following discussion will be framed around the specific questions themselves.

a. The diabatic cross-equatorial flow

Near the equator at the solstices it is believed that the mean meridional flow is basically cross-equatorial from summer to winter hemisphere. In an overall sense this flow may be understood as a dynamic response to the distribution of heat sources and sinks—specifically, the broad, weak ozone summer heating, and the locally intense CO_2 cooling in the polar night. At upper levels (at and above the stratopause) a very strong flow is generated because of the low densities there.

Evidence for this circulation is not available directly from observations, since it is all but impossible to construct honest zonal mean profiles of meridional wind. However, there is indirect evidence from two sources: first, the "rhodium 102" tracer experiment related by Kalkstein (1962) and, second, the numerical models with realistic radiation codes. The former set of observations strongly supported the pole-to-pole circulation concept; although it did not give precise measurements of this flow, it did suggest, say, that a 1 m s^{-1} magnitude would not be unreasonable, in view of the sudden increases in rhodium in high latitudes at 6-month intervals. Numerical models, second, are able to give more precise estimates based on diabatic circulation expectations; again the magnitude of 1 m s^{-1} at mesospheric heights is very common (e.g., Holton and Wehrbein, 1980) with perhaps a local maximum in $\bar{v}$ near the equator of order 3 m s^{-1}.

Now in view of the possible inertial instability of the middle atmosphere near the equator, there is first of all an uncertainty concerning such a local maximum in meridional wind. If inertial rolls such as those of Fig. 1 are present, it is obvious that we cannot interpret the expected diabatic cross-equatorial flow as being locally true anywhere; instead, it must be interpreted as a "net cross-equatorial mass flow", i.e., as a vertical integral of the mean motion.

However, even this interpretation is contingent upon another unproven assumption, namely, that the inertial instability has no "macroscopic" effect on the motion of the extratropical middle atmosphere. In mathematical terms the question is whether or not a small hyperbolic or parabolic region should effect the outer elliptical solution for the mean meridional streamfunction. At this time the only relevant result seems to be that of Hunt (1981), who in spite of his inertial instability nevertheless observed an adequately strong mesospheric jet at high latitudes. Previous numerical models are of little help here since none of them actually simulated any inertial instability. In the following, we shall assume that the expected diabatic cross-equatorial flow is unaffected by the presence of the instability, although future studies will be required to confirm this.

b. Numerically-predicted zonal mean motions

The diabatic circulation discussed above, however, raises what is the most famous and disturbing enigma of the dynamics of the middle atmosphere, namely, the inability of such a strong meridional flow to generate a realistic easterly–westerly jet structure, particularly at upper levels. In fact, the Coriolis torques associated with a 1 m s^{-1} flow can readily produce mesospheric zonal motions more than an order of magnitude larger than those observed!

In the last two decades we have been customarily rescued from this dilemma by Leovy's (1964) *tour de force* "Rayleigh friction" concept. Leovy found that by postulating a linear drag on the mean zonal flow, with a rate coefficient approximately an order of magnitude larger than the annual cycle frequency, a realistic jet structure could be produced. This drag is intended to parameterize the effect of breaking gravity waves on the mean flow. Since the gravity waves are generally very nearly steady with respect to the earth's surface, their net effect is to deposit momentum (specifically, *zero* momentum) at the upper levels where they are absorbed. In this overall sense, Leovy's postulate seemed (and still seems) quite reasonable.

A modest improvement on Leovy's idea was advanced by recent numerical modelers (Schoeberl and Strobel, 1978; Holton and Wehrbein, 1980) in which the "Rayleigh friction coefficient" was given a height dependence such that it is significant only in the mesosphere. This assumption is conceptually more reasonable than Leovy's constant coefficient, and also was found to result in realistic flows at upper levels.

Today, while theoreticians have come to appreciate Leovy's postulate as correct in some overall sense, they have also recognized that it should probably be treated as more of a theoretical expedient than as a physically realistic parameterization. This arises from the realization that internal gravity waves do not act locally in a manner consistent with a Rayleigh friction parameterization. The point of effective absorption may indeed be very difficult to determine since it may involve critical-level absorption (Booker and Bretherton, 1967), spontaneous interaction due to wave transience (Dunkerton, 1981), and nonlinear wave breaking (Lindzen, 1981). While in all of these cases the waves can act to decelerate the mean flow by a large amount, they do not act simply as $-\alpha_M \bar{u}$, so to speak. As Lindzen pointed out, for example, the gravity wave acceleration may actually in some cases *increase* as the critical level is approached—just the reverse of Rayleigh friction.

Now regardless of precisely how the gravity wave drag be parameterized, there seems to be no reason to doubt its importance, particularly in the extratropical mesosphere. However, for the purposes of this paper it is important to realize that by incorporating such a drag in a numerical model, there is a question raised as to how the inertial instability is affected by it. Obviously, the overall drag proposed by Leovy would dramatically reduce the instability of the flow, as will be evident a little later when we consider the stability of the Holton–Wehrbein (1980) model. However, while this is a conjecture on our part, we are of the belief that a mesospheric gravity wave drag on a local basis may not have such precedence over the inertial instability.

To illustrate this, it is helpful to conceive of two views of the middle atmosphere insofar as the equatorial inertial instability is concerned. One view, which might be labeled the conservative one, would regard the middle atmosphere as akin to the laboratory simulation of inertial instability, having an easterly–westerly jet structure forming "walls" analogous to those of the differentially rotating cylinder, which in the laboratory provided a generating mechanism for the instability. But by regarding the *observed* mesospheric jet structure as providing a destabilizing influence, we would find that such a configuration (complicated, of course, by the spherical geometry of the earth's atmosphere), while it would provide an unstable flow near the equator, it would be a *weakly* unstable one, compared to what we might have otherwise expected from the diabatic circulation.

On the other hand, the liberal viewpoint would have a different base, taking instead as its starting point the diabatic circulation and the associated mean Coriolis torques. Obviously, the instability of the flow would now be much increased by this assumption, by an order of magnitude or more. In this view the inertial instability and gravity wave drag would have equal *a priori* emphasis. We are inclined to regard this view as the correct one, as the overall drag concept proposed by Leovy is not valid on a local basis (Lindzen, 1981).

Two other facts of interest to this discussion are as follows. First, Lindzen (1981) has suggested that in tropical latitudes the dominant eddy diffusion mechanism originates in the breaking diurnal tide, which occurs at ~85 km, leading to a layer of enhanced eddy diffusion and wave-induced deceleration above this level. Implicit in Lindzen's remarks is the belief that gravity waves are dominant poleward of 30° or so. These waves, on the other hand, generate wave-induced diffusion and deceleration (which, incidentally, are separate effects as Lindzen notes) at much lower levels (i.e.,

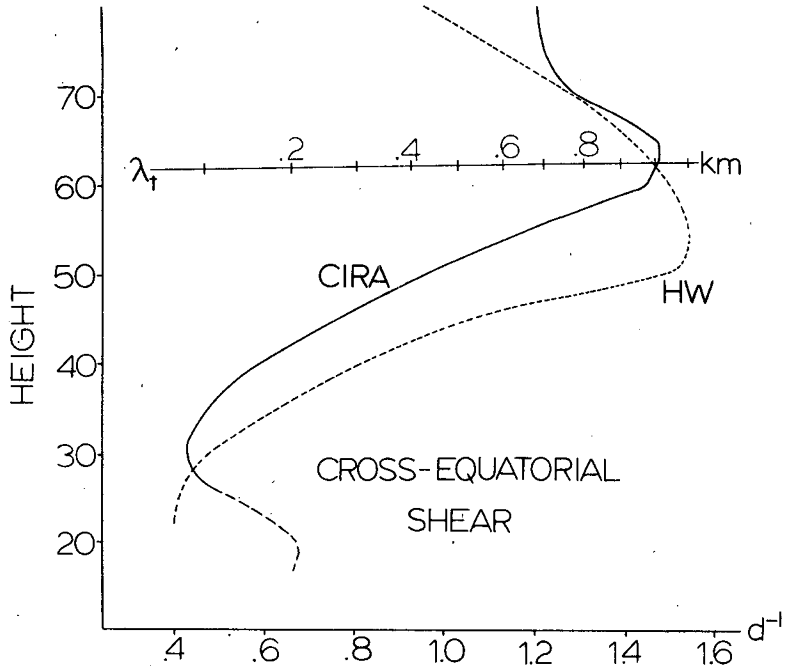

FIG. 2. "Observed" (solid line) and predicted (dashed line) cross-equatorial shear due to the diabatic circulation. The latter are taken from Holton and Wehrbein's numerical model, and result from a meridional flow of strength 3 m s^{-1} or less, coupled with a significant mesospheric drag coefficient. Also shown is the necessary vertical wavelength λ_t of a symmetric inviscid disturbance which would result in growth, given the observed and predicted shears.

in the mesosphere) with some important seasonal variations.

Secondly, the reader will note that Hunt's (1981) numerical model did not contain any kind of gravity wave-drag parameterization. Hence the results of his model, with its overly strong mesospheric jet, can be interpreted as being something more like what the atmosphere is trying to attain *via* the Coriolis torque apart from gravity wave-induced decelerations.

c. Observed cross-equatorial shears

In Fig. 2 is plotted, first of all, the observed cross-equatorial shear, derived from CIRA (1965) by taking finite differences between 10°N and 10°S. Evident in the figure is a maximum shear of order 1.5 $days^{-1}$ in the mesosphere.

Now because this shear is everywhere unstable in the inviscid problem, the question arises as to its legitimacy. Can this shear be tested for inertial stability, or more generally can any *observed* shear profile be unstable? Because this diagram is based on time-averaged data, taken from a sparse network of tropical stations, it is to be suspected that either this observed profile is already stabilized, partially or wholly, by inertial instabilities (together with diffusive processes), or, more speculatively, the profile in itself may be incorrect. Perhaps if the latter alternative be correct an analogy may exist in the tropospheric lapse rate—a monthly mean standard atmosphere characterized by a superadiabatic lapse rate would be virtually unthinkable; in comparison, a monthly mean inertially unstable shear may be equally unlikely.

Because of these limitations it is felt that this "observed" profile should have little value in a stability investigation of this sort, though it may provide a check on the degree to which the flow is stabilized by diffusive processes. In any case it is obvious that more observational efforts are badly needed to give a better assessment of the inertial stability of the region.

d. Eddy diffusion?

Added to the list of questions is the uncertainty over the role of eddy diffusion in inertial instability. It is apparent that eddy diffusion (in contrast to molecular diffusion) could have two distinct roles in the instability. On the one hand, an already existing eddy diffusion, arising, say, from the breaking internal gravity waves and tides of Lindzen (1981), may provide a stabilizing influence. Assuming that this diffusion may be parameterized as second-order diffusion, the development of Section 2 could then be used to ascertain the stability of the region. More generally we might envision this diffusion as preventing the appearance of scales of motion smaller than some critical vertical wavelength (i.e., a higher-order kind of diffusion). In any case the stabilizing influence of the eddy diffusion would be felt insofar as turbulent eddies could break up the inertial rolls, preventing parcels from travelling consistently in their predicted paths (Fig. 1), thereby effectively "diffusing" away the disturbance (the diffusion of heat would also be important).

On the other hand, it must be recognized that eddy diffusion may also be the *result* of an existing inertial instability! This will *always* be the case in a strongly unstable situation. Far from hindering the instability, this kind of eddy diffusion would be an integral part of it. Presumably, this turbulence would arise from the instability of the inertial rolls to smaller scales of motion.

e. Destabilizing role of thermal damping

Of considerable importance in the dynamics of the middle atmosphere is the radiative relaxation of temperature disturbances (e.g., Holton, 1975). The question arises as to the significance of this thermal damping for the equatorial inertial instability, especially since the thermal damping rate at the stratopause may equal or exceed the value of the Coriolis parameter near the equator.

Interestingly, it is found, however, that thermal damping is actually *destabilizing* in our case. This result was implicit in the above discussion when variations in Prandtl number were considered; a small σ (large thermal diffusivity but small viscosity) actually reduced the critical shear for instability. This result holds for more general kinds of thermal damping, though we omit the details here. Physically, this is due to the fact that the thermal damping would act to remove the temperature perturbations, which hydrostatically give rise to the perturbation pressure fields which *oppose* the meridional motions (Fig. 1). (An opposite conclusion would obtain in the vertical shear problem, however.)

f. Quantitative estimates of the inertial stability

Three major difficulties in assessing the inertial stability of the equatorial middle atmosphere identified above are 1) ascertaining the observed horizontal mean wind shear at the equator; 2) comparing this shear to what would be expected from the diabatic cross-equatorial mass flow; and 3) trying to assess the impact of the gravity wave drag incorporated in some numerical models. In determining the inertial stability of this region we might approach the question along these three lines. First, the stability of the observations and the model results from Holton and Wehrbein (1980) can be assessed. Secondly, comparison can be made to the stability of a flow induced directly by the diabatic circulation.

Before doing any of this, however, one last difficulty arises, which is the determination of the eddy diffusion to be used in the stability test. While the eddy diffusion coefficient K_{zz} has been calculated in the past, it must be recognized that regardless of how such a quantity is calculated, the procedures used in the calculation are arbitrary to some extent. For example, some methods are based on assumptions as to the mixing mechanisms (e.g., gravity waves), while other methods ascertain the value of K_{zz} by trying to reproduce observed tracer distributions.

A recent calculation of K_{zz} has been reported by Nastrom *et al.* (1980). Those authors used an *ad hoc* formula to calculate K_{zz} as being due to breaking gravity waves. Although their formula may be subject to question, we have chosen to display their eddy diffusivities because they agree in an overall sense with previous estimates, but actually indicate slightly higher values of K_{zz} in the tropical winter mesosphere than previously reported.

What is displayed in Fig. 3 is the diffusive damping rate νm_c^2 calculated from (2.6) and (2.7) using the K_{zz} $(=\nu)$ of Nastrom *et al.* (1980). Those authors attributed their somewhat larger tropical mesospheric K_{zz} (of order 200 m^2 s^{-1}) to the presence of Kelvin waves such as were observed by Hirota (1978). While it is this author's belief that Hirota's waves may actually be of importance in the semiannual oscillation (Dunkerton, 1979), and may therefore act in a *nondiffusive* way, we note that the general magnitude of K_{zz} (100 m^2 s^{-1} or more) is consistent with values quoted in Lindzen (1981) who derived them in a different way. In any case the vertical variation of the diffusivity is roughly exponential, which appears reduced in our figure having been raised to the one-fifth power.

Also shown for the sake of comparison are the thermal damping rate and Rayleigh friction coefficient used in Holton and Wehrbein (1980). Both of these quantities appear small when compared to the viscous damping rate at the critical vertical wavenumber.

In Fig. 2 was also plotted the vertical variation of the cross-equatorial shear in Holton and Wehrbeins' model at the solstices. This shear is nicely consistent with that of CIRA (1965) in terms of the maximum amplitude, although the vertical distribution is shifted downward slightly. (This may be indicative of an excessive Rayleigh friction in the model mesosphere.)

The second-order viscous diffusion test may be applied to the observed and model shears shown in

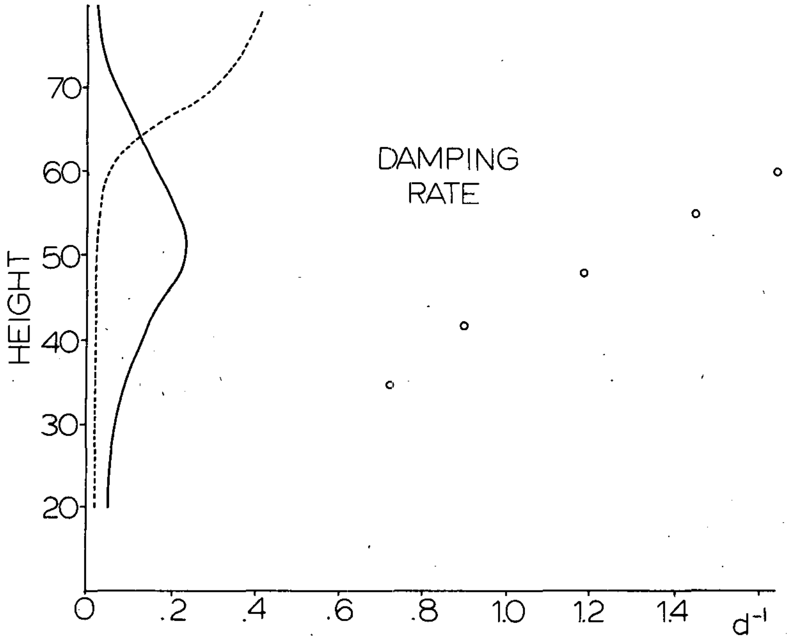

FIG. 3. Radiative relaxation rate (solid line), Rayleigh friction coefficient (dashed line), and viscous damping rate at the critical wavenumber (circles).

Fig. 2. The critical shear for instability is equal to the viscous damping rate times $2(5)^{1/2}$. We find that the observed and model shears are both stabilized by the K_{zz} values used here. In fact, a value of 0.1 $m^2 s^{-1}$ would be sufficient to stabilize the maximum horizontal shears.

Fig. 2 also indicates that the maximum possible vertical wavelength for instability is quite small, being only 1–1½ km in the mesosphere. The extremely small vertical wavelengths in the stratosphere are a definitive commentary on the stability of this region! Indeed, such finely structured rolls would be almost unimaginable.

On the other hand, the necessary mesospheric vertical wavelengths are certainly observable, in principle, but here the high values of eddy diffusion would seem to preclude the instability in these shear profiles. We note in passing that the thermal relaxation, though it is destabilizing, is insufficient to alter our conclusions, since it would reduce the critical shear by only a few percent, given these values of K_{zz}.

Incidentally, the information in Fig. 2 reveals why the Holton–Wehrbein (1980) model, like its predecessors, has not found the inertial instability. (Because this model did not contain a vertical diffusion, there is no reason why inertial instability should not be observed.) The reason is simply that the vertical grid spacing is insufficient to allow growing modes. In fact, a shear almost four times as large would be required in the Holton–Wehrbein model before instability would set in, given their grid spacing of 5 km. We believe that Hunt's (1981) recent simulation contained the instability because of a coincidence of two circumstances: the relatively fine grid (2.5 km) and a mesospheric jet over twice that observed.

The above remarks have established that assuming the eddy diffusion is stabilizing, the "observed" and model shears are both inertially stable. These remarks, however, have not conclusively decided the stability of the real atmosphere for two reasons.

First, as already stated the eddy diffusion may be the *result* of inertial instability, rather than a hindrance of it. While it is impossible at this time to prove that eddy diffusion in the equatorial mesosphere is the result of inertial instability, we feel that it is equally difficult to prove the second alternative.

Second, supposing that the eddy diffusion *is* stabilizing, there remains an enormous discrepancy between the observed or model shears, and those which would be expected from the diabatic circulation. It is interesting to ask what cross-equatorial flow would generate (*via* the Coriolis torque alone) a horizontal shear surpassing even its largest critical value (associated with a 200 $m^2 s^{-1}$ diffusion). Given a sinusoidal annual cycle it is found that a meridional flow of only 0.75 $m s^{-1}$ is sufficient. Our point is simply this: that the expected diabatic cross-equatorial flow is much larger than this, and therefore trying to bring the atmosphere into a strongly unstable configuration above the stratopause. Given such a large value of diffusion, the vertical wavelengths of the inertial instability may exceed those indicated in Fig. 3. Hence it may be necessary to account for vertical variations in diffusion, density, shear, etc., in a more complete treatment. However, it is also possible that eddy diffusion in the tropical mesosphere may be the result of inertial instability, in which case the values shown in Fig. 3 would provide a logical starting point.

4. Conclusion

On the basis of the foregoing discussion it would seem that while a theory of equatorial inertial instability is not difficult to formulate, the practical application of the theory is complicated by the fact that in the atmosphere there is a great difference between the observed zonal mean wind distribution, and that predicted by the diabatic circulation. There is undoubtedly a complicated interaction between the gravity wave drag, the breaking gravity wave eddy diffusion, and the tendency for equatorial inertial instability as air parcels cross the equator. Unfortunately, Leovy's (1964) frictional drag scheme would seem to obscure the picture, rather than help it. There is clearly a need to incorporate a more realistic gravity wave drag parameterization in order to judge the likelihood of inertial instability accurately.

Hopefully, this paper will help to provide both observational and theoretical impetus to the entire question. In conclusion, a few words will be said concerning the relevance of this subject.

First, the occurrence of inertial instability would have some effect on the easterly phase of the semiannual zonal wind oscillation insofar as this phase owes its origin to horizontal advection (Dunkerton, 1979). Holton and Wehrbein (1980), for example, found such an oscillation due to this effect. Lest it be thought that inertial instability would prevent such an oscillation, it should be noted that a redistribution of momentum so as to stabilize the flow with respect to inertial instability would not inhibit the easterly cycle; rather, it would reduce its magnitude. In Fig. 4 this has been schematically drawn; indicated in the figure is the fact that a stable, parabolic wind profile in the (formerly) unstable hemisphere, drawn so as to give equal areas above and below the unstable linear shear profile, would predict a reduction in the strength of the easterly flow at the equator. In the stable hemisphere, a strong gradient of zonal wind would appear to result. (In the figure, the dashed line would take into account the momentum transport associated with our neutral viscous disturbance shown in Fig. 1.)

The redistribution of momentum shown in the figure is hypothetical, and must necessarily involve some departures from conservative motion. However, it is clearly evident that an inertial instability

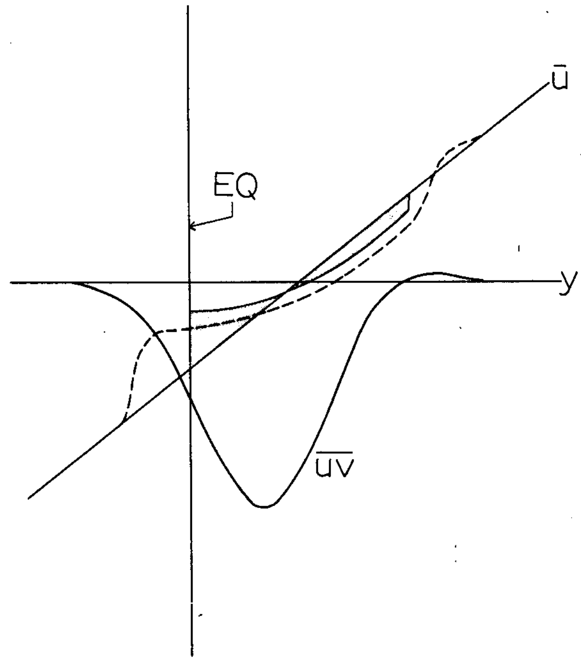

FIG. 4. An illustration of how inertial instability would redistribute momentum in the horizontal so as to conserve it, while at the same time requiring the final mean flow to be inertially stable. A better estimate is provided by the dashed line. A reduction of strength in the semiannual wind component of 50% or more is observed.

of the sort discussed here would have some effect on the easterly phase of the semiannual oscillation.

A second area of importance concerns the appearance of the tropical mesosphere. Instead of being characterized by a simple monotonic cross-equatorial flow at the solstices at all heights, the inertial instability would contribute to a great deal of irregularity in the vertical distribution of meridional velocity. While observations have given some indication of irregularity of this sort, they have not been formerly studied with inertial instability in mind.

Finally, the *magnitude* of the instability would be important insofar as it would determine whether the instability is basically laminar, dominated by diffusion, or is more turbulent, *resulting* in enhanced eddy diffusion. Because of the contamination of data with gravity waves and tidal motions at mesospheric heights, it may be difficult to discern this difference. But in any case it is hoped that this study will provide some stimulus for observational efforts along this line.

Acknowledgments. This research was supported by the Atmospheric Research Section, National Science Foundation, Grant ATM79264-87. Thanks go to J. R. Holton for timely advice during the preparation of this work.

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