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OPTIMUM EXPULSION OF BRINE FROM SEA ICE

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Abstract. A simple model is constructed to exploit the concept of drainage "trees" as observed during the process of expulsion for desalination of sea ice. Linear analysis is used together with a consistent approximation procedure for solution of the system to determine critical minimum Rayleigh numbers for the onset of instability. It is found that the system functions optimally when there is both a finite angle of tilt of the drainage channel as well as no tilt at all. The determination of the nonzero angle depends critically on the lateral mean density gradient, and numerical values for critical angles of inclination to the vertical obtained agree quantitatively with those reported.

Introduction

Over a period of many years, several proposals have been presented in an attempt to explain the means whereby sea ice desalinates. Untersteiner [1968] has outlined and described the salient proposals and his assessment includes (1) brine pocket diffusion, (2) flushing, (3) gravity drainage, and (4) expulsion. All of these mechanisms play a role in the ultimate removal because, in view of the supporting evidence (particularly experiments and field observations both direct and related), it is difficult for any one to represent the sole mechanism. On the other hand, a most convincing argument for the process of expulsion in combination with gravity drainage can be made in terms of efficiency and completeness. N. Untersteiner (private communication, 1983) especially supports this process when the ice is thin. As the ice thickens and survives at least one summer, then a shift to flushing seems more relevant.

Basically, the expulsion process centers around a brine pocket imbedded in the ice as it cools. Some brine will freeze around the edges of the pocket, but since the ice will occupy more space than the water from which it formed, pressure builds in the pocket. Then, the ice can crack along a line of natural weakness, much as a crystallographic basal plane, and the cracks can focus on an already existing ice drainage channel. It is then possible to provide hydrostatic communication between different regions of the ice. Lake and Lewis [1970] have lucidly outlined this mechanism. The central idea has, however, a longer history. Bennington [1963] and Nakaya [1956] both discussed the basic concept of expulsion, and Bennington [1963] already conceived and observed brine drainage "trees", a coinage that is compatible with the expulsion-gravity drainage combination.

Once the hydrostatic connection between the ice and seawater has been formed, gravity drainage becomes important in removing the brine from the ice. Salty brine in the ice builds up because of capillary retention and, once a critical density gradient is exceeded, the onset of convection can overturn the water in the ice, replacing the brine with lower salinity water. Subsequently, brine reaccumulates until convection sets in once again. Observational evidence for this type of process is given to support the Lake and Lewis argument as well where the data indicate cyclic convection overturn in the first few centimeters of ice above the water-ice interface.

Another observation of the Lake-Lewis work should also be noted. A series of ice drainage channels similar to the dendritic drainage pattern of a river and its tributaries make up the primary outlets and are typically 1 in length in 3 to 4 thick ice. Feeding into these vertical tubes at angles of 40° to 54° from the vertical are much smaller capillary-sized channels. In other words, the mechanism is not simply a direct downward movement but rather a combination labyrinth of the inclined feeder channels and larger vertical drainage ports.

Convective motion can be treated by stability theory. Since there is no mean velocity, the stability of the equilibrium between gravity acting in the mean density gradient and the viscous retardation combined with diffusion is the basic issue and gravity drainage is possible. Such analysis has been made in tubes and channels in several cases with varying geometry [cf. for example, Pellew and Southwell, 1940; Taylor, 1954; Wooding, 1959; Yih, 1959]. The theory clearly predicts the fundamental process for the overturning motion for vertical or horizontal alignment of the channels. Modeling the physics of the ice drainage in this manner raises, however, the important question as to whether or not there is an intrinsic reason for an angle of inclination preference. Generalization of the stability problem is the purpose of this work. It will be seen that the combination of a vertical-inclined network indeed lends to optimal drainage and expulsion.

Stability Model

The basic model consists of a long capillary-sized drainage channel with rectangular cross section and inclined to the vertical. The channel is considered filled with incompressible fluid of constant viscosity and diffusivity but with variable density.

Perturbing the system and analyzing the stability makes use of the following points: (1) linear techniques, (2) principle of exchange of stabilities (i.e., at the onset of instability the most unstable mode is from a time independent system) [Pellew and Southwell, 1940; Yih, 1959], (3) channel is assumed to be very long so that

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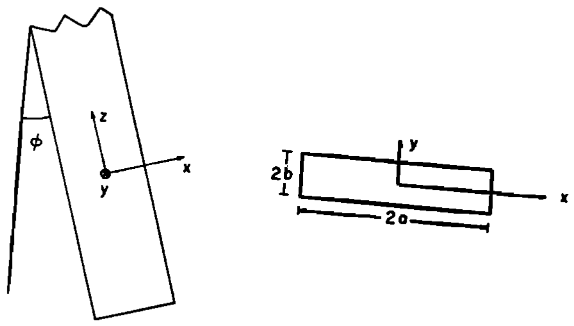

Fig. 1. (a) Channel schematics; (b) channel cross section.

end effects can be neglected, (4) fluid motion is incompressible according to the Boussinesq approximation.

By using a Cartesian coordinate system as shown in Figure 1 with z measured positively up the tube and the tilt angle (ϕ) of the tube measured from the vertical, the steady linearized governing equations for the perturbations can be developed. The momentum components are

$$\begin{aligned} -\frac{\partial p}{\partial x} - g \sin\phi \; \theta + \nu \nabla^2 u &= 0 \\ -\frac{\partial p}{\partial y} \qquad\qquad + \nu \nabla^2 v &= 0 \\ -\frac{\partial p}{\partial z} - g \cos\phi \; \theta + \nu \nabla^2 w &= 0 \end{aligned} \tag{1}$$

where u, v, w are the x, y, z components of the disturbance velocity; p is the disturbance pressure divided by a reference density; θ is the disturbance density divided by the same reference density; g is the acceleration due to gravity; ν is kinematic coefficient of viscosity; ∇^2 is the Laplacian operator.

The incompressibility condition is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{2}$$

and the equation for the diffusion of mass is

$$u \frac{\partial \overline{\Theta}}{\partial x} + w \frac{\partial \overline{\Theta}}{\partial z} = D\nabla^2 \theta \tag{3}$$

where $\overline{\Theta}$ is the mean density divided by the reference value and D is the diffusivity coefficient. It should be noted from (3) that the axes choice eliminates any effect on the average density surface in the y direction owing to the tilting, and, therefore, $\overline{\Theta}$ is a function of x and z only.

The problem is complete by enforcing the boundary conditions. These can be written as

$$u = v = w = \partial\theta/\partial n = 0 \tag{4}$$

at the boundary walls where $\partial/\partial n$ means normal derivative of the fluctuating density. This is sufficient to specify completely the problem as posed and mathematically results in eigenvalue calculations.

An equivalent stability problem for a vertical cylindrical tube has been solved by Taylor [1954], Yih [1959], and Wooding [1959]. Yih has also considered the plane boundary or rectangular geometry problem. The system of equations (1)-(3) reduces immediately to these cases by allowing $\phi = 0$ and Θ to be a function of z, the height only. In general, this problem is considerably less complicated than that with an arbitrary angle of tilt. There are numerous details that can be ascertained for any of these limiting situations, but the most important is the calculation of a minimum value of the Rayleigh number, the nondimensional measure of gravity versus viscous-diffusion effects, for the onset of instability.

As can be surmized, a direct analytical solution for the inclined channel problem is not as feasible. Instead, a rational approximation procedure is sought that allows for tractability and still maintains the critical physics involved. This can be done by carefully examining what has been found. The work of Lake and Lewis [1970, cf. Figure 8, p. 594] clearly shows that, when the feeder arms form in the cracks in the sea ice, then the channel has a characteristic low aspect ratio or simply $b < a$, using the notation of Figure 1. Interpretation of this fact in terms of the channel model not only means that b is the important length scale but implies that changes of any of the dependent variables will be primarily in the y direction. The velocity, v, in this direction is, however, small when compared to the components u or w because the major motion must be gravity driven and therefore largely in the x-z plane.

Two additional points can be derived from the stability analysis made by Yih [1959] for the vertical planar flow. First, the mean density gradients that appear in equation (3) can be taken as constants or simply a bilinear variation of the mean density in terms of x and z. Within the framework of the approximations this appears sufficient and certainly is adequate to represent the tilted density surface being acted upon by gravity. Second, dependence of the perturbations on z, the coordinate along the length of the channel, can be neglected. Any variation in this direction would have to be periodic, since the channel is taken as infinite in length and no dependence is simply the constant term. Supporting this assumption is the result of Yih who found that the minimum Rayleigh number value is exactly such a solution for the vertical channel. Although not shown here, it is a consistent approximation in the inclined problem as well. (In a completely different physical problem but analogous mathematical formulation with inclination involved, Haaland and Sparrow [1973] also assumed independence of z at the outset. An argument is made to show that this input is tantamount to requiring neutral stability. It does not necessarily mean that other modes cannot exist but rather that any other solution will occur for larger values of the Rayleigh number.) The velocity w in the z direction remains finite.

Final prospective can be achieved by nondimensionalization of the system equations. Define new variables in such a way that all quantities will be of order unity. Specifically, let

$$X = (x/b); \; Y = P(y/b) \tag{5}$$

where P is a nondimensional number (to be determined) larger than 1 so that Y is of equal order with X.

The relevant velocity scale measure for comparison is based on the rate of diffusion and is defined as $q_o = D/b$. Nondimensional components become

$$U = Q(u/q_o) \ ; \ V = P(v/q_o) \ ; \ W = Q(w/q_o) \qquad (6)$$

where P is the same number as given in (5) and Q is a number less than 1 so that all components are scaled by the same rule for equal ordering.

The pressure need not be considered due to the predominance of the gravity driven motion. Hence, for purposes of the model all pressure gradients can be neglected at the outset. A posteriori confirmation will clearly show that this step recognizes that the perturbed motion is slow or, in non-dimensional terms, the Reynolds number, based on b and q_o, is much smaller than 1. The pressure can be scaled with the Reynolds number (as is done in lubrication theory) and the Rayleigh number, but this complicates this approximation with no great addition to the central physics. Such influence is left to higher order corrections. In any case, it is impossible to retain the pressure gradient in both the x and y directions to lowest order.

The system of equations in terms of the new variables becomes, after incorporation of all additional assumptions,

$$-\frac{R_aQ}{P^2}\sin\phi\ \theta + \frac{1}{P^2}\frac{\partial^2 U}{\partial X^2} + \frac{\partial^2 U}{\partial Y^2} = 0$$

$$\frac{1}{P^2}\frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2} = 0 \qquad (7)$$

$$-\frac{R_aQ}{P^2}\cos\phi\ \theta + \frac{1}{P^2}\frac{\partial^2 W}{\partial X^2} + \frac{\partial^2 W}{\partial Y^2} = 0$$

with

$$\frac{\partial U}{\partial X} + Q\frac{\partial V}{\partial Y} = 0 \qquad (8)$$

and

$$\frac{(sU+W)}{QP^2(1+s^2)^{1/2}} = \frac{1}{P^2}\frac{\partial^2\theta}{\partial X^2} + \frac{\partial^2\theta}{\partial Y^2} \qquad (9)$$

where

$$R_a = \frac{g\overline{\rho}_z(1+s^2)^{1/2}\,b^4}{\mu D} \equiv \text{Rayleigh number}$$

and $s = \overline{\rho}_x/\overline{\rho}_z$ is the density gradient ratio. From the definition of the Rayleigh number, it is to be noted that $\overline{\rho}_z(1+s^2)^{1/2}b$, where $\overline{\rho}$ is the mean density, has been established as the reference density throughout. In other words, the reference density surface is perpendicular to the direction of gravity. The choices for the parameters P and Q are made by requiring that the richest set of equations in the limits $P \to \infty$, $Q \to 0$ be determined. In short, the lowest order equations should retain the requisite physics. The suitable choice is

$$P = R_a^{1/4}/(1+s^2)^{1/8} \ ; \ Q = 1/R_a^{1/2}(1+s^2)^{1/4} \qquad (10)$$

Taking the limits, the basic system becomes

$$\frac{\partial^2\theta}{\partial Y^2} = sU + W$$

$$\frac{\partial^2 U}{\partial Y^2} = \sin\phi\ \theta \qquad (11)$$

$$\frac{\partial^2 W}{\partial Y^2} = \cos\phi\ \theta$$

with

$$\frac{\partial^2 V}{\partial Y^2} = 0 \qquad (12)$$

and

$$\frac{\partial U}{\partial X} = 0 \qquad (13)$$

The reduced set of equations has three immediate results: (1) equation (13) shows that the equations (11) are actually ordinary differential equations since U, W, and θ will all be independent of X and solutions might be termed local; (2) the solution in the x-z plane is completely uncoupled from any possible movement in the y direction; (3) since there is no pressure gradient in (12), $V \equiv 0$ is the only possible solution that satisfies the boundary conditions. Any derivation would appear at higher order.

Combining the equations for U, W, and θ from (11) gives the single equation for the disturbance density

$$\frac{d^4\theta}{dY^4} - (s\sin\phi + \cos\phi)\ \theta = 0 \qquad (14)$$

Since (14) is actually an ordinary differential equation to this order, the solutions are

$$\theta = Ae^{\sigma Y} + Be^{-\sigma Y} + Ce^{i\sigma Y} + De^{-i\sigma Y} \qquad (15)$$

with $\sigma = (s\sin\phi + \cos\phi)^{1/4}$. Recalling the boundary conditions from (4) in terms of the new variables it is necessary that $d\theta/dY = d^2\theta/dY^2 = 0$ at $Y = \pm R_a^{1/4}/(1+s^2)^{1/8}$. Performing these operations with (15) yields an eigenvalue problem and results in finding the roots for the expression

$$\text{Cosh}2\tilde{\sigma}\ \text{Cos}2\tilde{\sigma} - 1 = 0 \qquad (16)$$

where

$$\tilde{\sigma} = R_a^{1/4}\ (s\sin\phi + \cos\phi)^{1/4}/(1+s^2)^{1/8}$$

Critical Equilibria

Calculation of the values that satisfy (16) is tantamount to determining a relation for the Rayleigh number, R_a, as a function of the tilt angle φ. The determination is not, however, explicit until the density ratio s is specified.

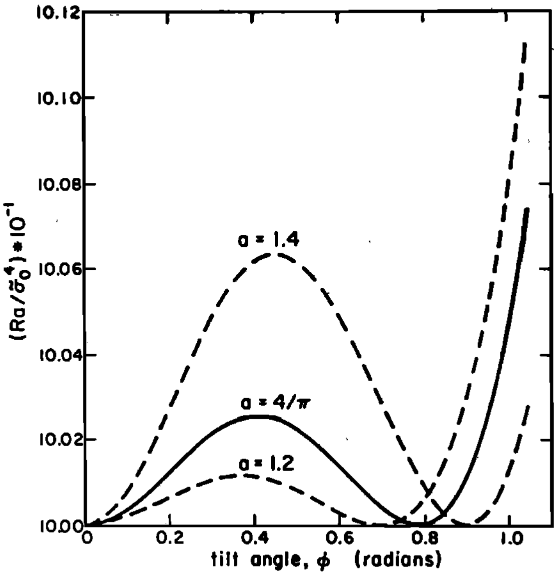

Figure 2

Fig. 2. Influence of tilt on critical Rayleigh number.

Although it has been assumed from the outset that the density gradient in the x and z direction, respectively, are constants, it cannot be ignored that the numerical value of this ratio will vary with varying angle of tilt. The limiting case of $\phi = 0$ clearly must mean that $s = 0$. Conversely, as the angle increases, then the cross gradient in the x direction must gain in importance. The simplest variation that incorporates these needs and remains consistent with the overall model is a linear function, or

$$s = a\phi \tag{17}$$

where the value of the slope, a, directly reflects the strength of the cross variation in the mean density gradient.

Returning to (16), the smallest root is the one of significance in the problem because of the interest in the minimum value of the Rayleigh number for the onset of instability. Defining this value to be $\tilde{\sigma}_0$, it is found that $\tilde{\sigma}_0 = 2.365$. Noting the definition for $\tilde{\sigma}$, (17) can be substituted, and explicit variation of the minimum Rayleigh number as a function of tilt becomes

$$R_a/\tilde{\sigma}_0^{\,4} = \frac{(1 + a^2\phi^2)^{1/2}}{(a\phi \sin\phi + \cos\phi)} \tag{18}$$

Plots of (18) are provided in Figure 2 and are presented as a result of two different criteria for the amplitude. The initial decision was to determine a so that $s = 1$ for $\phi = \pi/4$ or simply $a = 4/\pi$ by (17). Logically this carries the implication that the relative gradients are equal when the density surface is inclined at 45° from the vertical. The central curve shows the variation for the critical Rayleigh number with two clear minima at $\phi = 0$ and $\phi = 0.79$ radians (45.26°). In fact the values of $R_a/\tilde{\sigma}_0^{\,4}$ are identical at these two locations and denote maximum instability.

The Lake-Lewis report indicates that the drainage feeder channels occur in the range of tilt between 40° and 54°. Guided by the remarkable result obtained by the equal-gradient value of a shown by the central curve, two other curves were drawn to assess the relative change of the gradient strengths. Again, there are two equal minima with $\phi = 0°$, 0.70 radians (40.12°) with $a = 1.2$ and $\phi = 0°$, 0.90 radians (51.57°) for $a = 1.4$; for calibration, the case $a = 4/\pi \approx 1.3$.

Systematically, the expression (18) can be analyzed to determine whether or not there are specific values for a and ϕ such that $R_a/\tilde{\sigma}_0^{\,4}$ is minimum. This determination is straightforward and requires $\tan\phi^* = a^*\phi^*$ where a^*, ϕ^* denote the critical point values. Thus, there will always be two minima at two distinct angles, with one always $\phi^* = 0$ unless $a^* = 0$. The slope variable, a, affects the shift of the nonzero ϕ^* to lesser or greater values depending upon whether or not the lateral mean density gradient is less than or greater than the gradient in the vertical. It should be noted that the minimum Rayleigh number at $\phi^* = 0$ agrees with the Yih calculation for the vertical plane channel, as it should (and, it should be recalled, the minimum value was for a solution independent of the coordinate along the length of the channel, i.e., z). Judging from the Lake-Lewis data, it appears that the system functions optimally with the combination of the two drainage patterns and that the cross density gradient is essential to the overall mechanism. An experiment based on these concepts would be welcome.

Solutions for the velocity components U, W can be obtained by substituting the expression for θ in the necessary equation from (11) and the solutions are straightforward. Determining the velocity components that are more compatible to the brine expulsion system requires, however, additional information and is expressible in terms of a constraint. From the standpoint of this model it is sufficient (at least to a reasonable approximation) to ensure that the drainage is hydrostatically connected to the sea with seawater replacing salty brine lost by drainage. Freezing or other complications are hardly expressible within the scope of this work. Hydrostatic connection means

$$\iint dA = 0 \tag{19}$$

where integration is a cross-sectional area of the channel. The component W is then (U can be determined in a similar way)

$$W = \frac{\cos\phi}{\sigma^2 \cos\tilde{\sigma}} \, (\cos\tilde{\sigma} \sinh\sigma Y + \cosh\tilde{\sigma} \sin\sigma Y) \tag{20}$$

up to a multiplicative constant. It is clear that W is an odd function in Y and (19) is satisfied.

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